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On Inversion of Weierstrass Transform and its Saturation Class

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The Weierstrass transform

$$f(x) = \int_{-\infty}^{\infty} k(x-y, 1)\,\phi(y)\,dy, \qquad \text{where } k(x,t) = (4\pi t)^{-1/2} e^{-x^2/4t}, \quad (1)$$

and its inversion formula (see [3], p. 189)

$$\lim_{t\to 1-} \int_{d-i\infty}^{d+i\infty} K(s-x,t) f(s)\,ds = \phi(x), \qquad \text{where } K(s,t) = \left(\frac{\pi}{t}\right)^{1/2} e^{s^2/4t}, \quad (2)$$

give rise to the Gauss–Weierstrass singular integral

$$G(x, 1-t, \phi) = \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} K(s-x,t) f(s)\,ds = \int_{-\infty}^{\infty} k(x-u, 1-t)\,\phi(u)\,du. \quad (3)$$

Saturation theorems for Gauss–Weierstrass singular integral were studied recently by M. Kozima and G. I. Sunouchi ([5], p. 153), E. Görlich ([14], p. 131), and before that, by several other authors, references to whom may be found in [1].

For some Banach spaces E of functions, it was found that $G(x, 1-t; \phi)$ is saturated of order $1-t$, that is: $\|G(\cdot, 1-t; \phi) - \phi(\cdot)\|_E = o(1-t)$ as $t \to 1-$, if and only if $\phi(x) = 0$ a.e., and $\|G(\cdot, 1-t; \phi) - \phi(\cdot)\|_E = O(1-t)$ as $t \to 1-$, if and only if $\phi(x) \in \mathscr{R} \subset E$, $\mathscr{R} \neq \{0\}$; also the saturation class $\mathscr{R}$ was characterized.

These results may be applicable for the inversion of Weierstrass transform, (2), since they enable us to determine, under certain restriction, how "fast" the inversion operator tends to ϕ, and for what functions it tends at that rate (the saturation class). However, to determine if $\phi \in \mathscr{R}$, we have to check if $\|G(\cdot, 1-t, \phi) - \phi(\cdot)\|_E = O(1-t)$, $t \to 1-$, but it is ϕ that we want to find and actually try to calculate as a limit of $G(x, 1-t, \phi)$. To avoid the loop arising if we start with the Weierstrass transform, we try to determine if $\phi \in \mathscr{R}$ directly from $f(x)$ in (1); in the process we obtain results on other classes of functions, interesting by themselves.

THEOREM 1. *Let E be $L_p(-\infty,\infty)$, $1 \leqslant p < \infty$, or $C_0(-\infty,\infty)$, and let $\phi \in E$. Then for $n = 1, 2, \ldots$, $\|G_n(\cdot, 1-t, \phi)\|_E = o(1)$, $t \to 1-$, if and only if $\phi = 0$ (in E). Also, $\|G_n(\cdot, 1-t, \phi)\|_E = 0(1)$, $t \to 1-$, if and only if $\phi \in \mathscr{R}_n \subset E$. Here*

$$G_n(x, 1-t, \phi) = \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}\left(\frac{\partial}{\partial s}\right)^n K(s-x, t) f(s)\, ds, \tag{4}$$

$f(s)$, $K(s,t)$ are defined by (1) *and* (2), *and $\mathscr{R}_n$ is defined as follows:*

(a) $\mathscr{R}_n = \{\phi : \phi \in L_1$ *and* $\phi^{(n-1)}(x) \in B.V.\}$ *when* $E = L_1$,

(b) $\mathscr{R}_n = \{\phi : \phi \in L_p$ *and* $\phi^{(n)}(x) \in L_p\}$ *when* $E = L_p, 1 < p < \infty$,

(c) $\mathscr{R}_n = \{\phi : \phi \in C_0$ *and* $\phi^{(n)}(x) \in L_\infty\}$ *when* $E = C_0$.

Recalling the result of [5], p. 153, we have:

Corollary. If $\phi \in L_p(-\infty,\infty)$, $1 \leqslant p < \infty$, or $\phi \in C_0(-\infty,\infty)$, then

$$\|G(\cdot, 1-t, \phi) - \phi(\cdot)\| = O(1-t), \qquad t \to 1-,$$

if and only if

$$\|G_2(\cdot, 1-t, \phi)\| = O(1), \qquad t \to 1-.$$

Remark. In Theorem 1 there is a requirement made directly on ϕ—to belong to E. But for $E = L_p$, $1 \leqslant p < \infty$, necessary and sufficient conditions, on $f(x)$, for ϕ to belong to L_p ([3], pp. 195–6) are known. When $E = C_0$, the objection raised is justified, since no representation theorem is known for $\phi \in C_0(-\infty,\infty)$ (not even for $\phi \in$ B.C. $(-\infty,\infty)$).

For the proof of our theorem, we need the following

LEMMA. *Let $\psi \in \mathscr{D}$ (the L. Schwartz space of test functions), and let*

$$G_n(x, 1-t, \psi) \tag{5}$$

$$= \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}\left(\frac{\partial}{\partial s}\right)^n K(s-x, t)\frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty} k(s-y, 1)\psi(y)\, dy\, ds.$$

Then for $E = L_p$, $1 \leqslant p < \infty$, or $E = C_0$,

$$\|G_n(\cdot, 1-t, \psi) - (-1)^n \psi^{(n)}(\cdot)\|_{E^*} = o(1), \qquad t \to 1-, \tag{6}$$

where E^* is the conjugate space of E.

Proof. Using Fubini's theorem, we have

$$G_n(x, 1-t, \psi) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \psi(y)\, dy \int_{d-i\infty}^{d+i\infty} \left(\frac{\partial}{\partial s}\right)^n K(s-x, t)\, k(s-y, 1)\, ds$$

$$= \int_{-\infty}^{\infty} \psi(y) \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} \left(-\frac{\partial}{\partial x}\right)^n K(s-x, t)\, k(s-y, 1)\, ds$$

$$= \int_{-\infty}^{\infty} \psi(y) \left(-\frac{\partial}{\partial x}\right)^n \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} K(s-x, t)\, k(s-y, 1)\, ds.$$

The last step is justified since, for $t < 1$, we can prove, following the method in ([3], p. 190) and [2], that the inner integrals converge absolutely, uniformly in x. Using Theorem 2.5 of ([3], p. 177), we have

$$G_n(x, 1-t, \psi) = \int_{-\infty}^{\infty} \left(-\frac{\partial}{\partial x}\right)^{(n)} k(x-y, 1-t)\, \psi(y)\, dy$$

$$= (-1)^n \int_{-\infty}^{\infty} k(x-y, 1-t)\, \psi^{(n)}(y)\, dy.$$

Using ([5], p. 148),

$$\left\| \frac{G_n(\cdot, 1-t, \psi) - (-1)^n \psi^{(n)}(\cdot)}{1-t} - c\psi^{(n+2)}(\cdot) \right\|_{E^*} = o(1), \qquad t \to 1-. \tag{7}$$

Actually, in [5], the norm used is that of E; however, the proof with E^*-norm is quite similar, and the lemma proved there is used with the norm of E^* rather than that of E. Using (7), one immediately obtains (6). One can also prove (6) directly for $E^* = L_p$, $1 < p \leqslant \infty$, and $E^* =$ N.B.V.; for the latter the relation

$$\|G_n'(\cdot, 1-t, \psi) - (-1)^n \psi^{(n+1)}(\cdot)\|_{L_1} = o(1)$$

is used.

Proof of Theorem 1. We first prove that $\|G_n(x, 1-t, \phi\|_E = o(1)$, $t \to 1-$, implies $\phi = 0$ in E (the opposite implication is trivial). If $\psi \in \mathscr{D}$, then ψ is in the dual space of E, and

$$I(t) = \int_{-\infty}^{\infty} G_n(x, 1-t; \phi)\, \psi(x)\, dx = o(1), \qquad t \to 1-.$$

We have also, following [5], using Fubini's theorem and our Lemma,

$$I(t) = \int_{-\infty}^{\infty} G_n(x, 1-t, \psi)\, \phi(x)\, dx = (-1)^n \int_{-\infty}^{\infty} \psi^{(n)}(x)\, \phi(x)\, dx + o(1), \qquad t \to 1-.$$

Hence, $\langle \phi^{(n)}, \psi \rangle = 0$ for every $\psi \in \mathscr{D}$, and therefore $\phi^{(n)} = 0$ in $\mathscr{D}'$, or, ϕ is a polynomial of degree n, and since it belongs to L_p, $1 \leqslant p < \infty$, or to C_0, $\phi = 0$.

To determine $\mathscr{R}_n$, for $\psi(x) \in \mathscr{D}$, we get by using our Lemma,

$$\int_{-\infty}^{\infty} G_n(x, 1-t, \phi)\, \psi(x)\, dx = \int_{-\infty}^{\infty} G_n(x, 1-t, \psi)\, \phi(x)\, dx$$
$$= (-1)^n \int_{-\infty}^{\infty} \psi^{(n)}(x)\, \phi(x) + o(1), \qquad t \to 1-.$$

Using $\|G_n(x, 1-t, \phi)\|_E = O(1)$ in case $E = L_p$, $1 < p < \infty$, we recall that a closed ball is weakly compact, and since $\psi \in L_q$ (because it belongs to $\mathscr{D}$),

$$\int_{-\infty}^{\infty} G_n(x, 1-t_\nu; \phi)\, \psi(x)\, dx = \int_{-\infty}^{\infty} f(x)\, \psi(x)\, dx + o(1), \qquad \nu \to \infty,$$

and $\phi^{(n)}(x) - f(x)) = 0$ in $\mathscr{D}'$; therefore $f \in L_p$ implies $\phi^{(n)}(x) \in L_p$.

The weak $*$ conditional capactness of a bounded set in B.V. $(-\infty, \infty)$ will yield the result in case $E = L_1$, following [5]. A bounded set in C_0 with the weak topology introduced by $\psi \in \mathscr{D}$, will be conditionally compact with limits in L_∞, which will yield our theorem in case $E = C_0$.

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